Fragmentation of compositions and intervals

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Abstract

The fragmentation processes of exchangeable partitions have already been studied by several authors. In this paper, we examine rather fragmentation of exchangeable compositions, that means partitions of $\mathbb{N}$ where the order of the blocks counts. We will prove that such a fragmentation is bijectively associated to an interval fragmentation. Using this correspondence, we then calculate the Hausdorff dimension of certain random closed set that arise in interval fragmentations and we study Ruelle's interval fragmentation.

Key Words. Interval fragmentation, exchangeable compositions.

A.M.S. Classification. 60 J 25, 60 G 09.

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1 Introduction

Random fragmentations describe an object which splits as time passes. Two types of fragmentation have received a special attention : fragmentation of partitions of $\mathbb{N}$ and mass-fragmentation, i.e. fragmentation on the space $\mathcal{S}^{\downarrow} = \{s_1 \geq s_2 \geq \ldots \geq 0, \sum_i s_i \leq 1\}$. Berestycki [3] has proved that for each homogeneous fragmentation process of exchangeable partitions, we can canonically associate a mass fragmentation. More precisely, according to the work of Kingman [16], we know that if $\pi = (\pi_1, \pi_2, \ldots)$ is an exchangeable random partition of $\mathbb{N}$ (i.e. the distribution of π is invariant under finite permutation of $\mathbb{N}$), the asymptotic frequency of block π_i, $f_i = \lim_{n\to\infty} \frac{Card(\pi_i \cap \{1,\ldots,n\})}{n}$, exists a.s. We denote by $(|\pi_i|^{\downarrow})_{i\in\mathbb{N}}$ the sequence $(f_i)_{i\in\mathbb{N}}$ after a decreasing rearrangement. If $(\Pi(t), t \geq 0)$ is a fragmentation of exchangeable partitions, then $(|\Pi_i(t)|^{\downarrow}_{i\in\mathbb{N}}, t \geq 0)$ is a mass fragmentation. Conversely, a fragmentation of exchangeable partitions can be constructed from a mass fragmentation via a "paintbox process".

One of our goal in this paper is to develop an analog theory for fragmentations of exchangeable compositions and interval fragmentations. The notion of composition structure has been introduced by Gnedin [14] ; roughly speaking, it can be thought of a partition where the order of the block counts. Gnedin proved a theorem analogous to Kingman's Theorem in the case of

exchangeable compositions : for each probability measure P that describes the law of a random exchangeable composition, we can find a probability measure on the open subset of [0,1], such that P can be recovered via a "paintbox process". This is why it seems very natural to look for a correspondence between fragmentations of compositions and interval fragmentations.

The first part of this paper develops the relation between probability laws of exchangeable compositions and laws of random open subsets, and its extension to infinite measures. Then we prove that there exists indeed a one to one correspondence between fragmentation of compositions and interval fragmentations. The next part gives some properties and characteristics of these processes and briefly presents how this theory can be extended to time-inhomogeneous fragmentations and self-similar fragmentations.

We then turn our attention to the estimation of the Hausdorff dimension of random closed sets which arise in an interval fragmentation. Finally, as an application of this theory, we study in Section 6.1 a well known interval fragmentation introduced by Ruelle [2, 9, 11, 18] and we give a description of its semi-group of transition.

2 Exchangeable compositions and open subsets of $]0,1[$

2.1 Probability measures

In this section, we define exchangeable compositions following Gnedin [14], and recall some useful properties.
For $n \in \mathbb{N}$, let $[n]$ be the set of integers $\{1,\dots,n\}$ and write $[\infty] = \mathbb{N}$.

Definition 2.1 *For $n \in \mathbb{N}$, a composition of $[n]$ is an ordered sequence of disjoint, non empty subsets of $[n]$, $\gamma = (A_1,\dots,A_k)$, with $\cup A_i = [n]$.*
We denote by $\mathcal{C}_n$ the set of composition of $[n]$.

Let $\rho_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ be the restriction of a composition of $[n]$ to a composition of $[n-1]$ and let $\mathcal{C}$ be the projective limit of $(\mathcal{C}_n, \rho_n)$. We endow $\mathcal{C}$ with the product topology, then it is a compact set.

We say that a sequence $(P_n)_{n\in\mathbb{N}}$ of measure on $(\mathcal{C}_n)_{n\in\mathbb{N}}$ is a consistent sequence of measures if, for all $n \geq 2$, P_{n-1} is the image of P_n by the projection ρ_n, i.e., for all $\gamma \in \mathcal{C}_{n-1}$, we have

$$P_{n-1}(\Gamma_{n-1} = \gamma) = \sum_{\gamma' \in \mathcal{C}_n : \rho_n(\gamma') = \gamma} P_n(\Gamma_n = \gamma).$$

By Kolmogorov theorem, such a sequence $(P_n)_{n\in\mathbb{N}}$ determines the law of a random composition of $\mathbb{N}$.

In the sequel, for $n \in \mathbb{N} \cup \{\infty\}$, $\gamma \in \mathcal{C}_n$ and $A \subset [n]$, γ_A will denote the restriction of γ to A. Hence, for $m \leq n$ and, $\gamma_{[m]}$ will denote the restriction of γ to $[m]$.

A random composition Γ of $\mathbb{N}$ is called exchangeable if for all $n \in \mathbb{N}$, for every permutation σ of $[n]$ and for all $\gamma \in \mathcal{C}_n$, we have :

$$\mathbb{P}(\Gamma_{[n]} = \gamma) = \mathbb{P}(\sigma(\Gamma_{[n]}) = \gamma),$$

where $\sigma(\Gamma_{[n]})$ the image of the composition $\Gamma_{[n]}$ by σ. Hence, given an exchangeable random composition Γ, we can associate a function defined on finite sequences of $\mathbb{N}$ by

$$\forall k \in \mathbb{N}, \forall n_1,\dots,n_k \in \mathbb{N}^k, p(n_1,\dots,n_k) = \mathbb{P}(\Gamma_{[n]} = (B_1,\dots,B_k)),$$

with $|B_i| = n_i$ and $n_1 + \ldots + n_k = n$. This function determines the law of Γ and is called the exchangeable composition probability function (ECPF) of Γ.

Notation 2.2 *Let Γ be a composition of $\mathbb{N}$. For $i, j \in \mathbb{N}^2$, we will use the following notation :*

- *$i \sim j$, if i and j are in the same block.*
- *$i \prec j$, if the block containing i is before the block containing j.*
- *$i \succ j$, if the block containing i is after the block containing j.*

Definition 2.3 *Let U be an open subset of $[0,1]$. We construct a random composition of $\mathbb{N}$ in the following way :*
Let us draw $(X_i)_{i\in\mathbb{N}}$ iid random variables with uniform law on $[0,1]$. Then we use the following rules :

- *$i \sim j$, if $i = j$ or if X_i and X_j belong to the same component interval of U.*
- *$i \prec j$, if X_i and X_j do not belong to the same component interval of U and $X_i < X_j$.*
- *$i \succ j$, if X_i and X_j do not belong to the same component interval of U and $X_i > X_j$.*

This defines a probability measure on $\mathcal{C}$ that we shall denote P^U ; the marginal of P^U on $\mathcal{C}_n$ will be denoted by P_n^U. If ν is a probability measure on $\mathcal{U}$, we denote by P^ν the law on $\mathcal{C}$ which marginals are :

$$P_n^\nu(\cdot) = \int_{\mathcal{U}} P_n^U(\cdot) d\nu(U).$$

Let $\mathcal{U}$ be the set of open subset of $]0,1[$. For $U \in \mathcal{U}$, let

$$\chi_U(x) = \min\{|x-y|, y \in U^c\},\ x \in [0,1],$$

where $U^c = [0,1]\backslash U$. We define also a distance on $\mathcal{U}$ by :

$$d(U,V) = ||\chi_U - \chi_V||_\infty.$$

It will be convenient to use the notation $\mathbf{1} =]0,1[$. The composition of $\mathcal{C}_n$ (resp. $\mathcal{C}$) with a single non empty block will be denoted by $\mathbf{1}_n$ (resp. $\mathbf{1}_\mathbb{N}$) and we will write $\mathcal{C}_n^*$ for $\mathcal{C}_n\backslash\{\mathbf{1}_n\}$.

Let us recall here two useful theorems from Gnedin [14] :

Theorem 2.4 *[14] Let Γ be an exchangeable random composition of $\mathbb{N}$, $\Gamma_{[n]}$ its restriction to $[n]$. Let $(n_1, \ldots, n_k)$ be the sequence of the block sizes of $\Gamma_{[n]}$ and $n_0 = 0$. Define $U_n \in \mathcal{U}$ by :*

$$U_n = \bigcup_{i=1}^k \left] \frac{n_{i-1}}{n}, \frac{n_i}{n} \right[.$$

Then U_n converges almost surely to a random element $U \in \mathcal{U}$. The conditional law of Γ given U is P^U.
As a consequence, if P be an exchangeable probability measure on $\mathcal{C}$, then there exists a unique probability measure ν on $\mathcal{U}$ such that $P = P^\nu$.

Hence, for each exchangeable composition Γ, we can associate an element of $\mathcal{U}$ which we will call asymptotic open set of Γ and denote U_Γ.
We shall also write $|\Gamma|^\downarrow$ for the decreasing sequence of the lengths of the interval components of U_Γ. More generally, for $U \in \mathcal{U}$, $|U|^\downarrow$ will be the decreasing sequence of the interval component lengths of U.

Let us notice that this theorem is the analogue of Kingman's Theorem for the representation of exchangeable partitions. Actually, let $\mathbf{Q}$ be an exchangeable probability measure on $\mathcal{P}_\infty$, the set of partition of $\mathbb{N}$ and let π be a partition with law $\mathbf{Q}$.

Kingman [16] has proved that each block of π has almost surely a frequency, i.e. if $\pi = (\pi_1, \pi_2, \ldots)$, then

$$\forall i \in \mathbb{N} \qquad f_i = \lim_{n\to\infty} \frac{\sharp\{\pi_i \cap [n]\}}{n} \qquad \text{exists } \mathbf{Q}\text{-a.s.}$$

One calls f_i the frequency of the block π_i. Therefore, for all exchangeable random partitions, we can associate a probability on $\mathcal{S}^\downarrow = \{s = (s_1, s_2, \ldots), s_1 \geq s_2 \geq \ldots \geq 0, \sum_i s_i \geq 1\}$ which will be the law of the decreasing rearrangement of the sequence of the partition frequencies.

Conversely, given a law $\tilde{\nu}$ on $\mathcal{S}^\downarrow$, we can construct an exchangeable random partition whose law of its frequency sequence is $\tilde{\nu}$ (cf. [16]) : we pick $s \in \mathcal{S}^\downarrow$ with law $\tilde{\nu}$ and we draw a sequence of independent random variables U_i with uniform law on $[0,1]$. Conditionally on s, two integers i and j are in the same block of Π iff there exists an integer k such that $\sum_{l=1}^{k} s_l \leq U_i < \sum_{l=1}^{k+1} s_l$ and $\sum_{l=1}^{k} s_l \leq U_j < \sum_{l=1}^{k+1} s_l$. We denote by $\rho_{\tilde{\nu}}$ the law of this partition (and by a slight abuse of notation, ρ_u denotes the law of the partition obtained with $\tilde{\nu} = \delta_u$). Kingman's representation Theorem states that any exchangeable random partition can be constructed in this way.

Let $\wp_1$ be the canonical application from the set of composition $\mathcal{C}$ to the set of partition $\mathcal{P}_\infty$ and $\wp_2$ the application from the set $\mathcal{U}$ to the set $\mathcal{S}^\downarrow$ which associates to an element U of $\mathcal{U}$ the decreasing sequence $|U|^\downarrow$. To sum up, we have the following diagram between probability measures on $\mathcal{P}_\infty$, $\mathcal{C}$, $\mathcal{S}^\downarrow$, $\mathcal{U}$:

$$\begin{tikzcd}
(\mathcal{C}, P^\nu) \arrow[r, leftrightarrow, "\text{Gnedin}"] \arrow[d, "\wp_1"'] & (\mathcal{U}, \nu) \arrow[d, "\wp_2"'] \\
(\mathcal{P}_\infty, \rho_{\tilde{\nu}}) \arrow[r, leftrightarrow, "\text{Kingman}"] & (\mathcal{S}^\downarrow, \tilde{\nu}).
\end{tikzcd}$$

2.2 Representation of infinite measures on $\mathcal{C}$

In this section, we show how Theorem 2.4 can be extended to the case of an infinite measure μ on $\mathcal{C}$ such that :

- μ is exchangeable.
- $\mu(\mathbf{1}_\mathbb{N}) = 0$.
- For all $n \in \mathbb{N}$, $\mu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) < \infty$.

A measure on $\mathcal{C}$ fulfilling this three properties will be called a "fragmentation measure". We will see in the sequel that such a measure can always be associated to a fragmentation process and conversely.

We will prove that we can decompose every fragmentation measure μ in two measures, one characterizing μ on the compositions with asymptotic open set $U_\gamma =]0,1[$, and the other on the complementary event. The measure on the event $U_\gamma =]0,1[$ is called erosion measure and the measure on the event $U_\gamma \neq]0,1[$ is called dislocation measure.

Definition 2.5 *A measure ν on $\mathcal{U}$ is called a dislocation measure if :*

$$\nu(\mathbf{1}) = 0, \qquad \int_{\mathcal{U}} (1 - s_1)\nu(dU) < \infty,$$

where s_1 is the length of the largest interval component of U.

In the sequel, for any ν measure on $\mathcal{U}$, we define the measure P^ν on $\mathcal{C}$ by

$$P^\nu(d\gamma) = \int_{\mathcal{U}} P^U(d\gamma)d\nu(U).$$

Notice that if ν is a dislocation measure, then $P^\nu(d\gamma)$ is a fragmentation measure. In fact, the measure P^ν is exchangeable since P^U is an exchangeable measure.
For $U \neq \mathbf{1}$, we have $P^U(\mathbf{1}_{\mathbb{N}}) = 0$, and as $\nu(\mathbf{1}) = 0$, we have also $P^\nu(\mathbf{1}_{\mathbb{N}}) = 0$.
We now have to check that $P^\nu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) < \infty$ for all $n \in \mathbb{N}$. Let us fix $U \in \mathcal{U}$. Set $|U|^\downarrow = s = (s_1, s_2, \ldots)$.

$$P^U(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) = 1 - \sum_{i=1}^{\infty} s_i^n \leq 1 - s_1^n \leq n(1 - s_1)$$

and so $P^\nu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) < \infty$.

We can now state the following theorem :

Theorem 2.6 *Let ϵ_i be the composition of $\mathbb{N}$, $(\{i\}, \{\mathbb{N} \setminus \{i\}\})$ and $\epsilon = \sum_i \delta_{\epsilon_i}$. Let ϵ'_i be the composition of $\mathbb{N}$, $(\{\mathbb{N} \setminus \{i\}\}, \{i\})$ and $\epsilon' = \sum_i \delta_{\epsilon'_i}$. These are two exchangeable measures on $\mathcal{C}$. If μ is a fragmentation measure, there exists $c_l \geq 0$, $c_r \geq 0$ and a dislocation measure ν such that :*

$$\mu = c_l\epsilon + c_r\epsilon' + P^\nu.$$

Besides, the restriction of μ to $\{\Gamma \in \mathcal{C},\ U_\Gamma = \mathbf{1}\}$ is $c_l\epsilon + c_r\epsilon'$ and the restriction to $\{\Gamma \in \mathcal{C},\ U_\Gamma \neq \mathbf{1}\}$ is P^ν.

Recall that in the case of fragmentation measure on partitions, Bertoin [4] proved the following result :
Let $\tilde{\epsilon}_i$ be the partition of $\mathbb{N}$, $\{\{i\}, \{\mathbb{N} \setminus \{i\}\}\}$ and define the measure $\tilde{\epsilon} = \sum_i \delta_{\tilde{\epsilon}_i}$. Let $\tilde{\mu}$ be an exchangeable measure on $\mathcal{P}_\infty$ such that $\mu(\mathbf{1}_{\mathbb{N}}) = 0$ and $\tilde{\mu}(\pi \in \mathcal{P}_\infty, \pi_n \neq \mathbf{1}_n)$ is finite for all $n \in \mathbb{N}$. Then there exists a measure $\tilde{\nu}$ on $\mathcal{S}^\downarrow$ such that $\tilde{\nu}(1) = 0$ and $\int_{\mathcal{S}^\downarrow}(1 - s_1)\nu(ds)$, and a nonnegative number c such that :

$$\tilde{\mu} = \rho_{\tilde{\nu}} + c\tilde{\epsilon}.$$

Notice that Theorem 2.6 is an analogous decomposition as in the case of fragmentation measure on compositions, except that, in this case, there is two coefficients of erosion, one characterizing the left erosion and the other the right erosion.

Proof. We adapt a proof due to Bertoin [4] for the exchangeable partition to our case.
Set $n \in \mathbb{N}$. Set $\mu_n = \mathbf{1}_{\{\Gamma_{[n]} \neq \mathbf{1}_n\}}\mu$, therefore μ_n is a finite measure. Let $\overrightarrow{\mu_n}$ be the image of μ_n by the n-shift, i.e. :

$$i \overset{\overrightarrow{\Gamma}^n}{\prec} j \Leftrightarrow i + n \overset{\Gamma}{\prec} j + n, \qquad i \overset{\overrightarrow{\Gamma}^n}{\sim} j \Leftrightarrow i + n \overset{\Gamma}{\sim} j + n, \qquad i \overset{\overrightarrow{\Gamma}^n}{\succ} j \Leftrightarrow i + n \overset{\Gamma}{\succ} j + n.$$

Then $\overrightarrow{\mu_n}$ is exchangeable since μ is and furthermore, it is finite measure. So, we can apply Theorem 2.4 :

$$\exists\,!\ \nu_n \text{ finite measure on } \mathcal{U} \text{ such that } \overrightarrow{\mu_n}(d\gamma) = \int_{\mathcal{U}} P^U(d\gamma)\nu_n(dU).$$

According to theorem 2.4, since $\overrightarrow{\mu_n}$ is an exchangeable finite measure, $\overrightarrow{\mu_n}$-almost every composition has an asymptotic open set and so μ_n-almost every composition has also an asymptotic open set, and as $\mu = \lim \uparrow \mu_n$, μ-almost every composition has also an asymptotic open set.
Besides we have :

$$\mu_n(n+1 \not\sim n+2 \mid U_\Gamma = U) = \overrightarrow{\mu_n}(1 \not\sim 2 \mid U_\Gamma = U) = P^U(1 \not\sim 2) = 1 - \sum {s_i}^2 \geq 1 - s_1.$$

So

$$\mu_n(n+1 \not\sim n+2) \geq \int (1-s_1)\nu_n(dU).$$

Set $\nu = \lim_{n\to\infty} \uparrow \nu_n$. Since

$$\mu_n(n+1 \not\sim n+2) \leq \mu(n+1 \not\sim n+2) \leq \mu(1 \not\sim 2) < \infty,$$

we have

$$\int (1-s_1)\nu(dU) < \infty.$$

Hence ν is a dislocation measure. Set $\gamma_k \in \mathcal{C}_k$.

$$\begin{aligned}
\mu(\Gamma_{[k]} = \gamma_k, U_\Gamma \neq \mathbf{1}) &= \lim_{n\to\infty} \mu(\Gamma_{[k]} = \gamma_k, \Gamma_{\{k+1,\ldots,k+n\}} \neq \mathbf{1}_n, U_\Gamma \neq \mathbf{1}) \\
&= \lim_{n\to\infty} \mu(\overset{\rightarrow n}{\Gamma}_{[k]} = \gamma_k, \Gamma_{[n]} \neq \mathbf{1}_n, U_\Gamma \neq \mathbf{1}) \\
&= \lim_{n\to\infty} \overrightarrow{\mu_n}(\Gamma_{[k]} = \gamma_k, U_\Gamma \neq \mathbf{1}) \\
&= \int_{\mathcal{C}^*} P^U(\Gamma_{[k]} = \gamma_k)\nu(dU).
\end{aligned}$$

Thus we have

$$\mu(\,\cdot\,, U_\gamma \neq \mathbf{1}) = \int P^U(\,\cdot\,)\nu(dU).$$

We now have to study μ on the event $\{U_\gamma = \mathbf{1}\}$.
Let $\tilde{\mu}$ be μ restricted to $\{1 \not\sim 2, U_\gamma = \mathbf{1}\}$. Let $\overrightarrow{\tilde{\mu}}$ be the image of $\tilde{\mu}$ by the 2-shift. The measure $\overrightarrow{\tilde{\mu}}$ is finite and exchangeable and its asymptotic open set is almost surely $\mathbf{1}$, so $\overrightarrow{\tilde{\mu}} = a\delta_{\mathbf{1}}$ where a is a nonnegative number.
So $\tilde{\mu} = c_1\delta_{\gamma_1} + \ldots + c_{10}\delta_{\gamma_{10}}$ where $\gamma_1, \ldots, \gamma_6$ are the six possible compositions build from the blocks $\{1\}$, $\{2\}$, $\mathbb{N}\backslash\{1,2\}$,
$\gamma_7 = (\{1\}, \mathbb{N}\backslash\{1\})$, $\gamma_8 = (\{2\}, \mathbb{N}\backslash\{2\})$,
$\gamma_9 = (\mathbb{N}\backslash\{1\}, \{1\})$, $\gamma_{10} = (\mathbb{N}\backslash\{2\}, \{2\})$. We must have $c_1 = \ldots = c_6 = 0$, for otherwise, by exchangeability, we would have $\mu(\{1\}, \{n\}, \mathbb{N}\backslash\{1,n\}) = c > 0$ and this would yield $\mu(S_2^*) = \infty$. By exchangeability, we also have $c_7 = c_8$ and $c_9 = c_{10}$ and so, by exchangeability,

$$\mu\mathbf{1}_{\{U_\gamma = \mathbf{1}\}} = c_l \sum_i \delta_{\epsilon_i} + c_r \sum_i \delta_{\epsilon'_i}. \square$$

As in section 2.1, we can now draw a diagram between fragmentation measures on $\mathcal{C}$ and $\mathcal{P}_\infty$ and dislocation measures on $\mathcal{U}$ and $\mathcal{S}^\downarrow$. Let us recall that $\wp_1$ is the canonical projection of $\mathcal{C}$ to

$\mathcal{P}_\infty$, and denote $q : (\mathcal{U}, \mathbb{R}_+, \mathbb{R}_+) \mapsto (\mathcal{S}^\downarrow, \mathbb{R}_+)$ the application defined by $q(U, a, b) = q(|U|^\downarrow, a+b)$. Then we have the following diagram :

$$\begin{tikzcd}
(\mathcal{C}, \mu) \arrow[r, leftrightarrow, "\text{Theorem 2.6}"] \arrow[d, "\wp_1"'] & \big(\mathcal{U}, (\nu, c_l, c_r)\big) \arrow[d, "q"'] \\
(\mathcal{P}_\infty, \tilde{\mu}) \arrow[r, leftrightarrow, "\text{Bertoin}"] & \big(\mathcal{S}^\downarrow, (\tilde{\nu}, c_l + c_r)\big).
\end{tikzcd}$$

It remains to prove that $\tilde{\mu} = \rho_{\tilde{\nu}} + (c_l + c_r)\tilde{\epsilon}$. Set $\tilde{\mu} = \rho_{\overline{\nu}} + c\tilde{\epsilon}$. Since $\tilde{\mu}$ is the image by $\wp_1$ of μ, we have

$$\tilde{\mu}(\tilde{\epsilon}_1) = \mu(\epsilon_1) + \mu(\epsilon'_1) \text{ and then } c = c_r + c_l.$$

Let us fix $n \in \mathbb{N}$ and $\pi \in \mathcal{P}_n \backslash \{\mathbf{1}_n\}$. Set $A = \{\gamma \in \mathcal{C}_n, \wp_1(\gamma) = \pi\}$. Remark now that for all $U, V \in \mathcal{U}$ such that $|U|^\downarrow = |V|^\downarrow$, we have $P^U(A) = P^V(A)$. Moreover we have $P^U(A) = \rho_s(\pi)$ if $s = |U|^\downarrow$. So

$$P^\nu(A) = \int_{\mathcal{S}^\downarrow} P^U(A)\nu(U, |U|^\downarrow = ds) = \int_{\mathcal{S}^\downarrow} \rho_s(\pi)\tilde{\nu}(ds) = \rho_{\tilde{\nu}}(\pi).$$

We get

$$\mu(A) = P^\nu(A) + c_l\epsilon(A) + c_r\epsilon'(A) = \rho_{\tilde{\nu}}(\pi) + (c_l + c_r)\tilde{\epsilon}(A) = \rho_{\overline{\nu}}(\pi) + (c_l + c_r)\tilde{\epsilon}(A) = \tilde{\mu}(\pi).$$

So we deduce that $\overline{\nu} = \tilde{\nu}$. □

3 Fragmentation of compositions and interval fragmentation

3.1 Fragmentation of compositions

Definition 3.1 *Let us fix $n \in \mathbb{N}$ and $\gamma \in \mathcal{C}_n$ with $\gamma = (\gamma_1, \dots, \gamma_k)$. Let $\gamma^{(\cdot)} = (\gamma^{(i)}, i \in \{1, \dots, n\})$ with $\gamma^{(i)} \in \mathcal{C}_n$ for all i. Set $m_i = \min \gamma_i$. We denote $\tilde{\gamma}^{(i)}$ the restriction of $\gamma^{(m_i)}$ to γ_i. So $\tilde{\gamma}^{(i)}$ is a composition of γ_i. We consider now $\tilde{\gamma} = (\tilde{\gamma}^{(1)}, \dots, \tilde{\gamma}^{(k)}) \in \mathcal{C}_n$.*
We denote by $FRAG(\gamma, \gamma^{(\cdot)})$ the composition $\tilde{\gamma}$. If $\gamma^{(\cdot)}$ is a sequence of i.i.d. random variables with law p, p-$FRAG(\gamma, \cdot)$ will denote the law of $FRAG(\gamma, \gamma^{(\cdot)})$.

We remark then that the operator $FRAG$ has some useful property. First, we have that $FRAG(\gamma, \mathbf{1}^{(\cdot)}) = \gamma$. Furthermore, the fragmentation operator is compatible with the restriction i.e. for every $n' \leq n$:

$$FRAG(\gamma, \gamma^{(\cdot)})_{[n']} = FRAG(\gamma_{[n']}, \gamma^{(\cdot)}).$$

Besides, the operator $FRAG$ preserves the exchangeability. More precisely, let $(\gamma^{(i)}, i \in \{1, \dots, n\})$ be a sequence of random compositions which is *doubly exchangeable*, i.e. for each i, $\gamma^{(i)}$ is an exchangeable composition, and moreover, the sequence $(\gamma^{(i)}, i \in \{1, \dots, n\})$ is also exchangeable. Let γ be an exchangeable composition of $\mathcal{C}_n$ independent of $\gamma^{(\cdot)}$. Then $FRAG(\gamma, \gamma^{(\cdot)})$ is an exchangeable composition. Let us prove this property. Let us fix a permutation σ of $[n]$. We shall prove that

$$FRAG(\gamma, \gamma^{(\cdot)}) \stackrel{law}{=} \sigma(FRAG(\gamma, \gamma^{(\cdot)})).$$

Let k be the number of blocks of γ and denote by $m_1, \dots, m_k$ the minimums of $\gamma_1, \dots, \gamma_k$. Let define now $m'_1, \dots, m'_k$ the minimums of $\sigma(\gamma_1), \dots, \sigma(\gamma_k)$. Define now $\gamma'^{(\cdot)} = (\gamma'^{(i)}, i \in \{1, \dots, n\})$ by

$$\gamma'^{(m'_i)} = \sigma(\gamma^{(m_i)}) \text{ for } 1 \leq i \leq k$$

$$\gamma'^{(j)} = \sigma(\gamma^{(f(j))}) \text{ for } j \in \{1,\dots,n\} \setminus \{m'_i, 1 \le i \le k\},$$

where f is the increasing bijection from $\{1,\dots,n\}\setminus\{m'_i, 1 \le i \le k\}$ to $\{1,\dots,n\}\setminus\{m_i, 1 \le i \le k\}$. We get

$$\sigma(FRAG(\gamma,\gamma^{(\cdot)})) = FRAG(\sigma(\gamma),\gamma'^{(\cdot)}).$$

Since $\sigma(\gamma) \overset{law}{=} \gamma$ and $\gamma'^{(\cdot)} \overset{law}{=} \gamma^{(\cdot)}$ and $\gamma'^{(\cdot)}$ remains independent of γ, we get

$$FRAG(\sigma(\gamma),\gamma'^{(\cdot)}) \overset{law}{=} FRAG(\gamma,\gamma^{(\cdot)}). \square$$

We can now define the notion of exchangeable fragmentation process of compositions.

Definition 3.2 *Let us fix $n \in \mathbb{N}$ and let $(\Gamma_n(t), t \ge 0)$ be a Markov process on $\mathcal{C}_n$ which is continuous in probability.*
We call Γ_n an exchangeable fragmentation process of compositions if :

- $\Gamma_n(0) = \mathbf{1}_n$ *a.s.*
- *Its semi-group is described in the following way : there exists a family of probability measures on exchangeable compositions $(P_{t,t'}, t \ge 0, t' > t)$ such that for all $t \ge 0, t' > t$ the conditional law of $\Gamma_n(t')$ given $\Gamma_n(t) = \gamma$ is the law of $P_{t,t'}$-$FRAG(\gamma,\gamma^{(\cdot)})$. The fragmentation is homogeneous in time if $P_{t,t'}$ depends only on $t'-t$.*

A Markov process $(\Gamma(t), t \ge 0)$ on $\mathcal{C}$ is called an exchangeable fragmentation process of compositions if, for all $n \in \mathbb{N}$, the process $(\Gamma_{[n]}(t), t \ge 0)$ is an exchangeable fragmentation process of compositions on $\mathcal{C}_n$.

In the sequel, a c-fragmentation will denote an exchangeable fragmentation process on compositions.

3.2 Interval fragmentation

In this section we recall the definition of a homogeneous[1] interval fragmentation [7].
We consider a family of probability measures $(q_{t,s}, t \ge 0, s > t)$ on $\mathcal{U}$. For all interval $I =]a,b[\subset]0,1[$, we define the affine transformation $g_I :]0,1[\to I$ given by $g_I(x) = a + x(b-a)$. We still denote g_I the induced map on $\mathcal{U}$, so, for $V \in \mathcal{U}$, $g_I(V)$ is an open subset of I. We define then $q^I_{t,s}$ as the image of $q_{t,s}$ by g_I. Hence $q_{t,s}$ is a probability measure on the open subset of I. Finally, for $W \in \mathcal{U}$ with interval decomposition $(I_i, i \in \mathbb{N})$, $q^W_{t,s}$ is the distribution of $\cup X_i$ where the X_i are independent random variables with respective law $q^{I_i}_{t,s}$.

Definition 3.3 *A process $(U(t), t \ge 0)$ on $\mathcal{U}$ is called a homogeneous interval fragmentation if it is a Markov process which fulfills the following properties :*

- *U is continuous in probability and $U(0) = \mathbf{1}$ a.s.*
- *U is nested i.e. for all $s > t$ we have $U(s) \subset U(t)$.*
- *There exists a family $(q_{t,s}, t \ge 0, s > t)$ of probability measure on $\mathcal{U}$ such that :*

$$\forall t \ge 0,\ \forall s > t,\ \forall A \subset \mathcal{U}, \quad \mathbb{P}(U(s) \in A |\ U(t)) = q^{U(t)}_{t,s}(A).$$

[1] In [7], Bertoin defines more generally self-similar interval fragmentations with index α. Here, the term homogeneous means that we only consider the case $\alpha = 0$.

In the following, we abbreviate an interval fragmentation process as an i-fragmentation.

We remark that if we take the decreasing sequence of the sizes of the interval components of an i-fragmentation, we obtain a mass-fragmentation, denoted here a m-fragmentation (see [4] for definition of m-fragmentation). But, with the m-fragmentation, we loose the genealogical aspect present in the i-fragmentation.

3.3 Link between i-fragmentation and c-fragmentation

From this point of the paper and until Section 4.4, the fragmentation processes will always be homogeneous in time, i.e. $q_{t,s}$ depends only on $s-t$, hence we will just write q_{t-s} to denote $q_{t,s}$.

Theorem 3.4 *There is a one to one correspondence between laws of i-fragmentations and laws of c-fragmentations. More precisely :*

- *let $(U(t), t \geq 0)$ be an i-fragmentation. Let $(V_i)_{i \geq 0}$ be a sequence of independent random variables uniformly distributed on]0,1[. Using the same process as in Definition 2.3 with $U(t)$ and $(V_i)_{i \geq 1}$, we define a process $(\Gamma(t), t \geq 0)$ on $\mathcal{C}$. Then $(\Gamma(t), t \geq 0)$ is a c-fragmentation and we have $U_{\Gamma(t)} = U(t)$ a.s. for each $t \geq 0$.*

- *Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation. Then $(U_{\Gamma(t)}, t \geq 0)$ is an i-fragmentation.*

Proof. We begin by proving the first point. We have by Theorem 2.4, $U_{\Gamma(t)} = U(t)$ a.s. for each $t \geq 0$. Let us fix $n \in \mathbb{N}$ and $t \geq 0$. We are going to prove that, for $s > t$, the conditional law of $\Gamma_{[n]}(s)$ given $\Gamma_{[n]}(t) = (\gamma_1, \ldots, \gamma_k)$ is the law of $FRAG(\Gamma_{[n]}(t), \gamma^{(\cdot)})$, where $\gamma^{(\cdot)}$ is a sequence of iid exchangeable compositions with law $\Gamma_{[n]}(s-t)$. Since $(U(s), s \geq 0)$ is a fragmentation process, we have $U(t+s) \subset U(t)$. By construction of $\Gamma_{[n]}(t)$, it is then clear that $\Gamma_{[n]}(t+s)$ is a finer composition than $\Gamma_{[n]}(t)$. Hence each singleton of $\Gamma_{[n]}(t)$ remains a singleton of $\Gamma_{[n]}(t+s)$. So we can assume that $\Gamma_{[n]}(t)$ has no singleton. For $1 \leq i \leq k$, fix $l \in \gamma_i$ and define

$$a_i = \sup\{a \leq V_l, a \notin U(t)\}, \qquad b_i = \inf\{b \geq V_l, b \notin U(t)\}.$$

Notice that a_i and b_i do not depend on the choice of $l \in \gamma_i$. Furthermore, since $\Gamma_{[n]}(t)$ has no singleton, we have $a_i < b_i$ almost surely. We define also

$$Y_j^i = \left(\frac{V_j - a_i}{b_i - a_i}\right)_{j \in \gamma_i, 1 \leq i \leq k}.$$

By construction of $\Gamma_{[n]}(t)$, the random variables $(Y_j^i)_{j \in \gamma_i, 1 \leq i \leq k}$ are independent and uniformly distributed on $]0,1[$. Besides, $(]a_i, b_i[)_{1 \leq i \leq k}$ are k distinct interval components of $U(t)$. Since $U(t)$ is a fragmentation process, the processes

$$\left(U^i(s) = \frac{1}{b_i - a_i}(U_{]a_i,b_i[}(s) - a_i), s \geq t\right)_{1 \leq i \leq k}$$

are k independent i-fragmentations with law $(U(s-t), s \geq t)$. Let $\gamma^{(i)}(s)$ be the composition of γ_i obtained from $U^i(s)$ and $(Y_j^i)_{j \in \gamma_i}$ using Definition 2.3. Hence, $\gamma^{(i)}(s)$ has the law of $\Gamma_{\gamma_i}(s-t)$ and the processes $(\gamma^{(i)}(s), s \geq t)_{1 \leq i \leq k}$ are independent. Furthermore, by construction we have $\Gamma_{[n]}(t+s) = FRAG(\Gamma_{[n]}(t), \gamma^{(\cdot)}(s))$. Hence, $(\Gamma_{[n]}(t), t \geq 0)$ has the expected semi group of transition.

Let us now prove the second point. In the following, we will write U_t to denote $U_{\Gamma(t)}$. First, we prove that for all $s > t$, $U_s \subset U_t$. Fix $x \notin U_t$, we shall prove $x \notin U_s$. We have

$\chi_{U_t}(x) = \min\{|x-y|, y \in U_t^c\} = 0$. Let U_t^n be the open subset of $]0,1[$ corresponding to $\Gamma_{[n]}(t)$ as in Theorem 2.4. So we have $\lim_{n\to\infty} d(U_t^n, U_t) = 0$. Fix $\varepsilon > 0$. Hence, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, $\chi_{U_t^n}(x) \leq \varepsilon$. This implies that :

$$\forall n \geq N, \exists y_n \notin U_t^n \text{ such that } |y_n - x| \leq \varepsilon.$$

Besides, as $(\Gamma(t), t \geq 0)$ is a fragmentation, we have for all $n \in \mathbb{N}$, $U_s^n \subset U_t^n$. Hence, we have also

$$\forall n \geq N, y_n \notin U_s^n,$$

and so $\chi_{U_s^n}(x) \leq \varepsilon$ for all $n \geq N$. We deduce that $\chi_{U_s}(x) = 0$ i.e. $x \notin U_s$.

We now have to prove the branching property. Fix $t > 0$. We consider the decomposition of U_t in disjoint intervals :

$$U_t = \coprod_{k\in\mathbb{N}} I_k(t).$$

Set $F_k(s) = U_{t+s} \cap I_k(t)$. We want to prove that, given U_t :

- $\forall l \in \mathbb{N}, \forall m_1, \ldots, m_l$ distinct, $F_{m_1}, \ldots, F_{m_l}$ are independent processes.
- F_k has the following law :

  $\forall A$ open subset of $]a,b[$, $\mathbb{P}((F_k(s), s \geq 0) \in A \,|I_k(t) =]a,b[) = \mathbb{P}((U_s, s \geq 0) \in (b-a)A + a)$.

For all $k \in \mathbb{N}$, there exists $i_k \in \mathbb{N}$ such that, if $J_{i_k}^n(t)$ denotes the interval component of U_t^n containing the integer i_k, then $J_{i_k}^n(t) \overset{n\to\infty}{\longrightarrow} I_k(t)$. Let B_k be the block of $\Gamma(t)$ containing i_k. As B_k has a positive asymptotic frequency, it is isomorphic to $\mathbb{N}$. Let f be the increasing bijection from the set of element of B_k to $\mathbb{N}$. Let us re-label the elements of B_k by their image by f. The process $(U_{\Gamma_{B_k}(t+s)}, s \geq 0)$ has then the same law as $(U_s, s \geq 0)$ and is independent of the rest of the fragmentation. Besides, given $I_k(t) =]a,b[$, $F_k(s) = a + (b-a)U_{\Gamma_{B_k}(t+s)}$, so the two points above are proved. □

Hence, this result completes an analogous result due to Berestycki [3] in the case of m-fragmentations and p-fragmentation (i.e. fragmentations of exchangeable partitions). We can again draw a diagram to represent the link between the four kinds of fragmentation :

$$\begin{tikzcd}
\big(\mathcal{C}, (\Gamma(t), t \geq 0)\big) \arrow[r, leftrightarrow, "\text{Theorem 3.4}"] \arrow[d, "\wp_1"'] & \big(\mathcal{U}, (U_{\Gamma(t)}, t \geq 0)\big) \arrow[d, "\wp_2"'] \\
\big(\mathcal{P}_\infty, (\Pi(t), t \geq 0)\big) \arrow[r, leftrightarrow, "\text{Berestycki}"] & \big(\mathcal{S}^\downarrow, (|U_{\Gamma(t)}|^\downarrow, t \geq 0)\big).
\end{tikzcd}$$

4 Some general properties

In this section, we gather general properties of i and c-fragmentations. Since the proof of these results are simple variations of those in the case of m and p-fragmentations [4], we will be a bit sketchy.

4.1 Measure of a fragmentation process

Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation. As in the case of p-fragmentation [4], for $n \in \mathbb{N}$ and $\gamma \in \mathcal{C}_n^*$, we define a jump rate from $\mathbf{1}_n$ to γ :

$$q_\gamma = \lim_{s\to 0} \frac{1}{s} \mathbb{P}\left(\Gamma_{[n]}(s) = \gamma\right).$$

With the same arguments as in the case of p-fragmentation, we can also prove that the family $(q_\gamma, \gamma \in \mathcal{C}_n^*, n \in \mathbb{N})$ characterizes the law of the fragmentation (you just have to use that distinct blocks evolve independently and with the same law). Furthermore, observing that we have

$$\forall n < m, \quad \forall \gamma' \in \mathcal{C}_n^*, \quad q_{\gamma'} = \sum_{\gamma \in \mathcal{C}_m, \gamma_{[n]} = \gamma'} q_\gamma,$$

and that

$$\forall n \in \mathbb{N}, \quad \forall \sigma \in \boldsymbol{\sigma}_n, \quad \forall \gamma \in \mathcal{C}_n^*, \quad q_\gamma = q_{\sigma(\gamma)},$$

we deduce that there exists a unique exchangeable measure μ on $\mathcal{C}$ such that $\mu(\mathbf{1}) = 0$ and $\mu(\mathcal{Q}_{\infty,\gamma}) = q_\gamma$ for all $\gamma \in \mathcal{C}_n^*$ and $n \in \mathbb{N}$, where $\mathcal{Q}_{\infty,\gamma} = \{\gamma' \in \mathcal{C}, \gamma'_{[n]} = \gamma\}$. Furthermore, the measure μ characterizes the law of the fragmentation.

We remark also that if μ is the measure of a fragmentation process, we have for all $n \in \mathbb{N}$,

$$\mu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) = \sum_{\gamma \in \mathcal{C}_n^*} q_\gamma < \infty.$$

So we can apply Theorem 2.6 to μ and we deduce the following corollary :

Corollary 4.1 *Let μ be the measure of a c-fragmentation. Then there exist a dislocation measure ν and two nonnegative numbers c_l and c_r such that :*

- $\mu \mathbf{1}_{\{U_\gamma \neq \mathbf{1}\}} = P^\nu$.
- $\mu \mathbf{1}_{\{U_\gamma = \mathbf{1}\}} = c_l \epsilon + c_r \epsilon'$.

With a slight abuse of notation, we will write sometimes in the sequel that $\mu = (\nu, c_l, c_r)$ when $\mu = P^\nu + c_l \epsilon + c_r \epsilon'$.

4.2 The Poissonian construction

Let us recall that we define in Section 2.2 a fragmentation measure as a measure μ on $\mathcal{C}$ such that :

- μ is exchangeable.
- $\mu(\mathbf{1}_\mathbb{N}) = 0$.
- For all $n \in \mathbb{N}$, $\mu(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) < \infty$.

Notice that if μ is the measure of a c-fragmentation, then μ is a fragmentation measure. Conversely, we now prove that, if we consider a fragmentation measure μ, we can construct a c-fragmentation with measure μ.

We consider a Poisson measure M on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$. Let M^n be the restriction of M to $\mathbb{R}_+ \times \mathcal{C}_n^* \times \{1, \ldots, n\}$. The intensity measure is then finite on the interval $[0, t]$, so we can order the atoms of M^n according to their first coordinate.
For $n \in \mathbb{N}$, $(\gamma, k) \in \mathcal{C} \times \mathbb{N}$, let $\Delta_n^{(.)}(\gamma, k)$ be the composition sequence of $\mathcal{C}_n$ defined by :

$$\Delta_n^{(i)}(\gamma, k) = \mathbf{1}_n \quad \text{if } i \neq k \qquad \text{and} \qquad \Delta_n^{(k)}(\gamma, k) = \gamma_{[n]}.$$

We construct then a process $(\Gamma_{[n]}(t), t \geq 0)$ on $\mathcal{C}_n$ in the following way :
$\Gamma_{[n]}(0) = \mathbf{1}_n$.
$(\Gamma_{[n]}(t), t \geq 0)$ is a pure jump process which jumps at times when an atom of M^n appears. More precisely, if (s, γ, k) is an atom of M^n, set $\Gamma_{[n]}(s) = FRAG(\Gamma_{[n]}(s^-), \Delta_n^{(.)}(\gamma, k))$.
We can check that this construction is compatible with the restriction ; hence, this defines a process $(\Gamma(t), t \geq 0)$ on $\mathcal{C}$.

Proposition 4.2 *Let μ be a fragmentation measure. The construction above of a process on compositions from a Poisson point process on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$, yields a c-fragmentation with measure μ.*

The proof is an easy adaptation of the Poissonian construction of p-fragmentations (cf. [4]). As the sequence $\Delta_n^{(.)}(\gamma, k)$ is doubly exchangeable, we also have that $\Gamma_{[n]}(t)$ is an exchangeable composition for each $t \geq 0$. Looking as the rate jump of the process $\Gamma_{[n]}(t)$, it is then easy to check that the constructed process is a c-fragmentation with measure μ.$\square$

A Poissonian construction of an i-fragmentation with no erosion is also possible with a Poisson measure on $\mathbb{R}_+ \times \mathcal{U} \times \mathbb{N}$ with intensity $dt \otimes \nu \otimes \sharp$. The proof of this result is not as simple as for compositions because we can not restrict to a discrete case as done above. In fact, to prove this proposition, we must take the image of the Poisson measure M above by an appropriate application. For more details refer to Berestycki [3] who have already proved this result for m-fragmentation and the same approach works in our case.

To conclude this section, let us notice how the two erosion coefficients affect the fragmentation. Let $(U(t), t \geq 0)$ be an i-fragmentation with parameter $(0, c_l, c_r)$. Set $c = c_l + c_r$. We have :

$$U(t) = \Big] \frac{c_l}{c}(1 - e^{-tc}), 1 - \frac{c_r}{c}(1 - e^{-tc}) \Big[\text{ a.s.}$$

Indeed, consider a c-fragmentation $(\Gamma(t), t \geq 0)$ such that $u_{\Gamma(t)} = U(t)$ a.s. We define $\mu_{c_l, c_r} = c_l \epsilon + c_r \epsilon'$. Hence $(\Gamma(t), t \geq 0)$ is a fragmentation with measure μ_{c_l, c_r}. Recall that the process $(\Gamma(t), t \geq 0)$ can be constructed from a Poisson measure on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu_{c_l, c_r} \otimes \sharp$. By the form of μ_{c_l, c_r}, we remark then that, for all $t \geq 0$, $\Gamma(t)$ have only one block non reduced to a singleton. Furthermore, for all $n \in \mathbb{N}$, the integer n is a singleton at time t with probability $1 - e^{-tc}$, and, given n is a singleton of $\Gamma(t)$, $\{n\}$ is before the infinite block of $\Gamma(t)$ with probability c_l/c and after with probability c_r/c. By the law of large number, we deduce that the proportion of singletons before the infinite block of $\Gamma(t)$ is almost surely $\frac{c_l}{c}(1 - e^{-tc})$ and the proportion of singletons after the infinite block of $\Gamma(t)$ is almost surely $\frac{c_r}{c}(1 - e^{-tc})$.

Remark 4.3 *Berestycki [3] has proved a similar result for the m-fragmentation. He also proved that if $(F(t), t \geq 0)$ is a m-fragmentation with parameter $(\nu, 0)$, then $\tilde{F}(t) = e^{-ct} F(t)$ is a m-fragmentation with parameter (ν, c). But, we can not generalize this result for i-fragmentation because the proportion of singleton between two successive component intervals of the fragmentation depends on the time where the two component intervals split.*

4.3 Projection from $\mathcal{U}$ to $\mathcal{S}^{\downarrow}$

We know that if $(U(t), t \geq 0)$ is an i-fragmentation, then its projection on $\mathcal{S}^{\downarrow}$, $(|U(t)|^{\downarrow}, t \geq 0)$ is a m-fragmentation. More precisely, we can express the characteristics of the m-fragmentation from the characteristics of the i-fragmentation.

Proposition 4.4 *The ranked sequence of the length of an i-fragmentation with measure* (ν, c_l, c_r) *is a m-fragmentation with parameter* $(\tilde{\nu}, c_l + c_r)$ *where* $\tilde{\nu}$ *is the image of* ν *by the application* $U \to |U|^{\downarrow}$.

Proof. Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation with measure $\mu = (\nu, c_l, c_r)$. Let $(\Pi(t), t \geq 0)$ be its image by $\wp_1$. The process $(\Pi(t), t \geq 0)$ is then a p-fragmentation. Set $n \in \mathbb{N}$ and $\pi \in \mathcal{P}_n^*$. We have

$$\begin{aligned} q_\pi &= \lim_{s \to 0} \frac{1}{s} \mathbb{P}(\Pi_{[n]}(s)) = \pi) \\ &= \lim_{s \to 0} \frac{1}{s} \mathbb{P}\left(\Gamma_{[n]}(s)) \in \wp^{-1}(\pi)\right) \\ &= \tilde{\mu}(\pi), \end{aligned}$$

where $\tilde{\mu}$ is the image of μ by $\wp_1$. Besides we have already prove that $\tilde{\mu} = (\tilde{\nu}, c_l + c_r)$. We consider now the i-fragmentation $(U_{\Gamma(t)}, t \geq 0)$ with measure (ν, c_l, c_r). We get that the process $(|U_{\Gamma(t)}|^{\downarrow}, t \geq 0)$ is a.s. equal to the m-fragmentation $(|\Pi(t)|^{\downarrow}, t \geq 0)$ which fragmentation measure is $(\tilde{\nu}, c_l + c_r)$. $\square$

According to Proposition 4.4 and using the theory of m-fragmentation (see [4]), we deduce then the following results :

- Let $(\Gamma(t), t \geq 0)$ be a c-fragmentation with parameter (ν, c_l, c_r). We denote by B_1 the block of $\Gamma(t)$ containing the integer 1. Set $\sigma(t) = -\ln |B_1(t)|$. Then $(\sigma(t), t \geq 0)$ is a subordinator. If we denote $\zeta = \sup\{t > 0, \sigma_t < \infty\}$, then there exists a non-negative function ϕ such that

  $$\forall q, t \geq 0, \;\; \mathbb{E}[\exp(-q\sigma_t), \zeta > t] = \exp(-t\phi(q)du).$$

  We call ϕ the Laplace exponent of σ and we have :

  $$\phi(q) = (c_l + c_r)(q+1) + \int_{\mathcal{U}} (1 - \sum_{i=1}^{\infty} |U_i|^{q+1}) \nu(dU),$$

  where $(|U_i|)_{i \geq 0}$ is the sequence of the lengths of the component intervals of U.

- An (ν, c_r, c_l) i-fragmentation $(U(t), t \geq 0)$ is proper (i.e. for each t, $U(t)$ has almost surely a Lebesgue measure equal to 1) iff

  $$c_l = c_r = 0 \text{ and } \nu\left(\sum_i s_i < 1\right) = 0.$$

4.4 Extension to the time-inhomogeneous case

We now briefly expose how the results of the preceding sections can be transposed in the case of time-inhomogeneous fragmentation. We will not always detail the proof since their are very similar as in the homogeneous case. In the sequel, we shall focus on c-fragmentation $(\Gamma(t), t \geq 0)$ fulfilling the following properties :

- for all $n \in \mathbb{N}$, let τ_n be the time of the first jump of $\Gamma_{[n]}$ and λ_n be its law. Then λ_n is absolutely continuous with respect to Lebesgue measure with continuous and strictly positive density.

- for all $\gamma \in \mathcal{C}_n^*$, $h_\gamma^n(t) = \mathbb{P}(\Gamma_{[n]}(t) = \gamma \mid \tau_n = t)$ is a continuous function of t.

Remark that a time homogeneous fragmentation always fulfills this two points. Indeed, in that case, λ_n is an exponential random variable and the function $h_\gamma^n(t)$ does not depend on t. As in the case of fragmentation of exchangeable partitions [2], for $n \in \mathbb{N}$ and $\gamma \in \mathcal{C}_n^*$, we can define an instantaneous rate of jump from $\mathbf{1}_n$ to γ :

$$q_{\gamma,t} = \lim_{s \to 0} \frac{1}{s} \mathbb{P}\left(\Gamma_{[n]}(\tau_n) = \gamma \;\&\; \tau_n \in [t, t+s] \mid \tau_n \geq t\right).$$

With the same arguments as in the case of fragmentation of exchangeable partitions [2], we can also prove that, for each $t > 0$, there exists a unique exchangeable measure μ_t on $\mathcal{C}$ such that $\mu_t(\mathbf{1}) = 0$ and $\mu_t(\mathcal{Q}_{\infty,\gamma}) = q_{\gamma,t}$ for all $\gamma \in \mathcal{C}_n \backslash \{\mathbf{1}\}$ and $n \in \mathbb{N}$, where $\mathcal{Q}_{\infty,\gamma} = \{\gamma' \in \mathcal{C}, \gamma'_n = \gamma\}$. Furthermore, the family of measure $(\mu_t, t \geq 0)$ characterizes the law of the fragmentation.

We remark also that if $(\mu_t, t \geq 0)$ is the family of measure of a fragmentation process, we have for all $n \in \mathbb{N}$,

$$\mu_t(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) = \sum_{\gamma \in \mathcal{C}_n^*} q_{\gamma,t} < \infty \text{ and } \int_0^t \mu_u(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\} = -\ln(\lambda_n(]t, \infty[)) < \infty.$$

So we can apply Theorem 2.6 to μ_t and we deduce the following proposition :

Corollary 4.5 *Let $(\mu_t, t \geq 0)$ be the family of measure of a c-fragmentation. Then there exists a family of dislocation measures $(\nu_t, t \geq 0)$ and two families of nonnegative numbers $(c_{l,t}, t \geq 0)$, $(c_{r,t}, t \geq 0)$ such that :*

- $\mu_t \mathbf{1}_{\{U_\pi \neq \mathbf{1}\}} = P^{\nu_t}$.
- $\mu_t \mathbf{1}_{\{U_\pi = \mathbf{1}\}} = c_{r,t}\epsilon + c_{r,t}\epsilon'$.

Besides we have for all $T \geq 0$,

$$\int_0^T \int_{\mathcal{U}} (1 - s_1)\, \nu_t\,(dU)\, dt < \infty \text{ and } \int_0^T (c_{l,t} + c_{r,t}) dt < \infty.$$

The first part of the proposition comes from Theorem 2.6. For the second part, use that

$$\int_{\mathcal{U}} (1 - s_1)\, \nu_t\,(dU) \leq \mu_t\left(\{\pi \in \mathcal{P}_\infty, \pi_{|2} \neq \mathbf{1}\}\right).$$

For the upper bound concerning the erosion coefficients, we remark that :

$$c_t + c'_t = \mu_t\left(\{1\}, \mathbb{N} \setminus \{1\}\right) + \mu_t\left(\mathbb{N} \setminus \{1\}, \{1\}\right). \square$$

In the same way as for homogeneous fragmentation, we define a fragmentation measure family as a family $(\mu_t, t \geq 0)$ of exchangeable measures on $\mathcal{C}$ such that, for each $t \in [0, \infty[$, we have :

- $\mu_t(\mathbf{1}_\mathbb{N}) = 0$.
- $\forall n \in \mathbb{N}$ $\mu_t(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) < \infty$ and $\int_0^t \mu_u(\{\gamma \in \mathcal{C}, \gamma_{[n]} \neq \mathbf{1}_n\}) du < \infty$.
- $\forall n \in \mathbb{N}$, $\forall A \subset \mathcal{C}_n^*$, $\mu_t(A)$ is a continuous function of t.

Proposition 4.6 *Let $(\mu_t, t \geq 0)$ be a fragmentation measure family. A c-fragmentation with fragmentation measure $(\mu_t, t \geq 0)$ can be constructed from a Poisson point process on $\mathbb{R}_+ \times \mathcal{C} \times \mathbb{N}$ with intensity $dt \otimes \mu_t \otimes \sharp$, where $\sharp$ is the counting measure on $\mathbb{N}$ in the same way as for time-homogeneous fragmentation.*

It is very easy to check that the proof of the homogeneous case applies here too.

Of course, a Poissonian construction of a time-inhomogeneous i-fragmentation with no erosion is also possible with a Poisson measure on $\mathbb{R}_+ \times \mathcal{U} \times \mathbb{N}$ with intensity $dt \otimes \nu_t \otimes \sharp$.

Concerning the law of the tagged fragment, if you define $\sigma(t) = -\ln |B_1(t)|$, with B_1 the block containing the integer 1, we have now that $\sigma(t)$ is a process with independent increments. And so, if we denote $\zeta = \sup\{t > 0, \sigma_t < \infty\}$, then there exists a family of non-negative functions $(\phi_t, t \geq 0)$ such that

$$\forall q, t \geq 0, \;\; \mathbb{E}[\exp(-q\sigma_t), \zeta > t] = \exp(-\int_0^t \phi_u(q) du).$$

We call ϕ_t the instantaneous Laplace exponent of σ at time t and we have :

$$\phi_t(q) = (c_{l,t} + c_{r,t})(q+1) + \int_{\mathcal{U}} (1 - \sum_{i=1}^{\infty} |U_i|^{q+1}) \nu_t(dU),$$

where $(|U_i|)_{i \geq 0}$ is the sequence of the lengths of the component intervals of U. Furthermore, an $(\nu_t, c_t, c_t')_{t \geq 0}$ i-fragmentation $(U(t), t \geq 0)$ is proper iff :

$$\forall t > 0, \quad c_{l,t} = c_{r,t} = 0 \quad \text{and} \quad \nu_t(\sum_i s_i < 1) = 0).$$

Finally, we can also compute the law of an $(0, c_{l,t}, c_{r,t})_{t \geq 0}$ i-fragmentation. After some calculus, we obtain that we have :

$$U(t) = \Big] \int_0^t c_{l,u} \exp(-C_u) du, \; 1 - \int_0^t c_{r,u} \exp(-C_u) du \Big[\text{ a.s.}$$

with $C_u = \int_0^u (c_{l,v} + c_{r,v}) dv$.

4.5 Extension to the self-similar case

A notion of self similar fragmentations has been also introduced [7]. We recall here the definition of a self similar p-fragmentation, the reader can easily adapt this definition to the three other cases of fragmentation.

Definition 4.7 *Let $\Pi = (\Pi(t), t \geq 0)$ be an exchangeable process on $\mathcal{P}_\infty$. We call Π a self similar p-fragmentation with index $\alpha \in \mathbb{R}$ if*

- $\Pi(0) = 1_{\mathbb{N}}$ *a.s.*
- Π *is continuous in probability*
- *For every $t \geq 0$, let $\Pi(t) = (\Pi_1, \Pi_2, \ldots)$ and denote by $|\Pi_i|$ the asymptotic frequency of the block Π_i. Then for every $s > 0$, the conditional distribution of $\Pi(t+s)$ given $\Pi(t)$ is the law of the random partition whose blocks are those of the partitions $\Pi^{(i)}(s_i) \cap \Pi_i$ for $i \in \mathbb{N}$, where $\Pi^{(1)}, \ldots$ is a sequence of independent copies of Π and $s_i = s|\Pi_i|^\alpha$.*

Notice that an homogeneous p-fragmentation corresponds to the case $\alpha = 0$.

We have still the same correspondence between the four types of fragmentation. In fact, a self similar fragmentation can be constructed from a homogeneous fragmentation with a time change :

Proposition 4.8 *[7] Let $(U(t), t \geq 0)$ be an homogeneous interval fragmentation with measure ν. For $x \in]0,1[$, we denote by $I_x(t)$ the interval component of $U(t)$ containing x. We define*

$$T_t^\alpha(x) = \inf\{u \geq 0, \int_0^u |I_x(r)|^{-\alpha} dr > t\} \text{ and } U^\alpha(t) = U(T_t^\alpha) = \bigcup I_x(T_t^\alpha(x)).$$

Then $(U^\alpha(t), t \geq 0)$ is a self similar interval fragmentation with index α.

A self similar i-fragmentation (or c-fragmentation) is then characterized by a quadruplet (ν, c_l, c_r, α) where ν is a dislocation measure on $\mathcal{U}$, c_l and c_r are two nonnegative numbers and $\alpha \in \mathbb{R}$ is the index of self similarity.

5 Hausdorff dimension of an interval fragmentation

Let $(U(t), t \geq 0)$ be a self similar i-fragmentation with index $\alpha > 0$. Let $K(t) = [0,1] \backslash U(t)$. The set $K(t)$ is a closed set, and if the fragmentation is proper (i.e. the fragmentation has with no erosion and its fragmentation measure verifies $\nu(\sum_i |U_i|^\downarrow < 1) = 0$), its Lebesgue measure is equal to 0. Hence, to evaluate the size of $F(t)$, we shall compute its Hausdorff measure. Here, we will just examine time-homogeneous fragmentation. First we recall the definition of the Hausdorff dimension of a subset of]0,1[.

Definition 5.1 *[13] Let $A \in]0,1[$. Let $d \geq 0$ and $r > 0$. We set*

$$J_d^r(A) = \inf \left\{ \sum_{i=1}^{\infty} |b_i - a_i|^d, A \subset \bigcup_{i=1}^{\infty} [a_i, b_i], |b_i - a_i| \leq r \right\} \text{ and } H_d(A) = \lim_{r \to 0^+} J_d^r(A),$$

(this limit exists since $J_d^r(A)$ decreases with r). $H_d(A)$ is the d-Hausdorff measure of A. Furthermore, there exists a unique number D such that

$$\forall d > D, H_d(A) = 0 \text{ and } \forall d < D, H_d(A) = \infty.$$

This number is the Hausdorff dimension of A and is denoted by $\dim_{\mathcal{H}}(A)$.

We will now calculate the Hausdorff dimension of the complement of a time-homogeneous i-fragmentation in the case where the measure of fragmentation fulfils some conditions.

Hypothesis 5.2 *Let ν be a dislocation measure. We assume that ν fulfills the following conditions :*

(H1) ν is conservative i.e. $\nu(\sum_i |U_i|^\downarrow < 1) = 0$.

(H2) There exists an integer k such that $\nu(|U_k|^\downarrow > 0) = 0$, i.e. ν is carried by the open sets with at most $k-1$ interval components.

(H3) Let $h(\varepsilon) = \int_{\mathcal{U}} (Card\{i, |U_i| \geq \varepsilon\} - 1) \nu(dU)$. Then h is regularly varying with index $-\beta$ as $\varepsilon \to 0+$.

(H4) *Let g be the left extremity of the largest interval component of a generic open set and d the right extremity. Then as $\varepsilon \to 0+$, we have either* $\liminf \frac{\nu(g\geq\varepsilon)}{\nu(d\leq 1-\varepsilon)} > 0$ *or* $\limsup \frac{\nu(g\geq\varepsilon)}{\nu(d\leq 1-\varepsilon)} < \infty$.

We can now state the theorem :

Theorem 5.3 *Let ν be a dislocation measure fulfilling Hypothesis 5.2. Let $(U(t), t \geq 0)$ be an i-fragmentation with characteristics $(\nu, 0, 0)$ and index of self-similarity α strictly positive. Let $K(t) = [0,1]\backslash U(t)$. Then the Hausdorff dimension of $K(t)$ is β for all $t > 0$ simultaneously, a.s.*

In fact, if the index of self-similarity is zero, the lower bound of the Hausdorff dimension still holds. Besides, Hypothesis $(H4)$ is only needed to prove the lower bound and allows a large class of dislocation measure such as symmetric measures or, at the opposite, measures for which the largest fragment is always on the same side.

Proof. We will first prove the upper bound. Let us recall a lemma proved by Bertoin in [8] for m-fragmentation processes whose dislocation measure fulfills Hypothesis 5.2.

Lemma 5.4 *[8] Let $(U(t), t \geq 0)$ be a self-similar $(\nu, 0, 0, \alpha)$ i-fragmentation with index of self similarity strictly positive and whose dislocation measure fulfills (H1), (H2), (H3). Let $(X(t) = (X_i(t))_{i\geq 1}, t \geq 0)$ the associated m-fragmentation. Let $N(\varepsilon, t) = Card\{i \geq 1, X_i(t) \geq \varepsilon\}$ and $M(\varepsilon, t) = \sum_i X_i(t)\mathbf{1}_{\{X_i\leq\varepsilon\}}$. Then $\lim_{\varepsilon\to 0+} \frac{N(\varepsilon,t)}{h(\varepsilon)}$ and $\lim_{\varepsilon\to 0+} \frac{M(\varepsilon,t)}{\varepsilon h(\varepsilon)}$ exist and are strictly positive and finite.*

Let us now fix $d \in]0,1[$ and look for a upper bound of the d-Hausdorff measure of $K(t)$. Let $I_\varepsilon =]0,1[\backslash\{$ interval components of $U(t)$ which size is larger than $\varepsilon\}$. So we have $K(t) \subset I_\varepsilon$ and $|I_\varepsilon| = M(\varepsilon, t)$ since ν is conservative. Furthermore, I_ε has at most $N(\varepsilon, t) + 1$ interval components. Using notation of Definition 5.1, we get :

$$J_d^\varepsilon(K(t)) \leq J_d^\varepsilon(I_\varepsilon) \leq \varepsilon^d \left(\frac{M(\varepsilon,t)}{\varepsilon} + N(\varepsilon,t) + 1\right) \leq h(\varepsilon)\varepsilon^d \left(\frac{M(\varepsilon,t)}{h(\varepsilon)\varepsilon} + \frac{N(\varepsilon,t)+1}{h(\varepsilon)}\right).$$

As h is regularly varying as $\varepsilon \to 0+$ with index $-\beta$, we deduce that for $d > \beta, h(\varepsilon)\varepsilon^d \to 0$ as $\varepsilon \to 0+$ and so $H_d(K(t)) = 0$. This proves that $\dim_{\mathcal{H}} K(t) \leq \beta$.

Let us now prove the lower bound. We first prove the lower bound for a homogeneous i-fragmentation, i.e. we suppose here that $\alpha = 0$. Let us fix $T_0 > 0$ and search for a lower bound of the Hausdorff dimension of $K(T_0)$. The two conditions of Hypothesis (H4) are symmetric by the transformation $x \to 1-x$, so, without loss of generality, we suppose here that $\liminf \frac{\nu(g\geq\varepsilon)}{\nu(d\leq 1-\varepsilon)} > 0$. Hence there exists a constant C such that for ε small enough we have $C\nu(g \geq \varepsilon) \geq \nu(d \leq 1-\varepsilon)$. We denote by $]g_t, d_t[$ the largest interval of the fragmentation at time t and $T = \inf\{t \geq 0, d_t - g_t \leq 1/2\} \wedge T_0$. So, for $0 < s < t < T$, $]g_t, d_t[\subset]g_s, d_s[$. The idea is to prove that $\dim_{\mathcal{H}}\{g_t, 0 < t < T\} \geq \beta$ and as $\{g_t, 0 < t < T\} \subset K(T_0)$, we will conclude that lower bound holds for $\dim_{\mathcal{H}} K(T_0)$.

We know that $(U(t), t \geq 0)$ can be constructed from a PPP on $\mathbb{R} \times \mathcal{U} \times \mathbb{N}$ with intensity measure $dt \times \nu \times \sharp$. So we have

$$g_t = \sum_{s\in\mathcal{D}\cap[0,t]} \xi_s(d_{s^-} - g_{s^-}),$$

where $(s, \xi_s)_{s \in \mathcal{D}}$ are the atoms of a Poisson measure on $\mathbb{R} \times [0,1]$ with intensity $ds \times \nu(g \in \cdot)$. We introduce now

$$\sigma_t = \sum_{s \in \mathcal{D} \cap [0,t]} \xi_s.$$

Then σ is a subordinator with Levy measure $\Lambda(d\varepsilon) = \nu(g \in d\varepsilon)$ and we have :

$$\forall\, 0 < s < t < T,\ g_t - g_s \geq \frac{1}{2}(\sigma_t - \sigma_s),$$

since $d_s - g_s < 1/2$ for $s \leq T$.

It is then well known that, if we want to prove that $\dim_{\mathcal{H}}\{g_t, 0 < t < T\} \geq \gamma$, it is sufficient to prove that g^{-1} is Hölder-continuous with exponent γ. We have then the following lemma :

Lemma 5.5 *Let $(f(t), 0 \leq t \leq T)$ and $(h(t), 0 \leq t \leq T)$ be two strictly increasing càdlàg functions such that for all $0 < s < t < T$, we have $h(t) - h(s) \geq \frac{1}{2}(f(t) - f(s))$. Define $f^{-1}(x) = \inf\{u \geq 0, f(u) > x\}$ and suppose that f^{-1} is Hölder-continuous with exponent γ. Then h^{-1} is also Hölder-continuous with exponent γ.*

Proof of the lemma. Let $s \geq t$ be two elements of the set $H = \{h(t), 0 \leq t \leq T\}$. Hence there exist $x \geq y$ such that $h(x) = s$ and $h(y) = t$. Then we have, for some constant K

$$h^{-1}(t) - h^{-1}(s) = y - x = f^{-1} \circ f(y) - f^{-1} \circ f(x) \leq K(f(y) - f(x))^{\gamma}.$$

Besides we have $\quad t - s = h(y) - h(x) \geq \frac{1}{2}(f(y) - f(x))$, so we get :

$$h^{-1}(t) - h^{-1}(s) \leq 2^{\gamma} K (t-s)^{\gamma}.$$

Furthermore, h^{-1} is constant on the interval components of H^c, and it follows then

$$h^{-1}(t) - h^{-1}(s) \leq 2^{\gamma} K (t-s)^{\gamma} \quad \text{for all } s < t. \quad \square$$

Hence to prove that $\dim_{\mathcal{H}}\{g_t, 0 < t < T\} \geq \beta$, we just have to prove that σ^{-1} is Hölder-continuous with exponent γ for all $\gamma < \beta$. We use then the following lemma :

Lemma 5.6 *[5] Let $(\sigma_s, s \geq 0)$ be a subordinator with no drift and Lévy measure Λ. Let $\Phi(\lambda) = \int_0^{\infty}(1 - e^{-\lambda x})\Lambda(dx)$ and $\gamma = \sup\{\alpha > 0, \lim_{\lambda \to \infty} \lambda^{-\alpha}\Phi(\lambda) = \infty\}$. Then, for every $\varepsilon > 0$, σ^{-1} is a.s. Hölder-continuous on compact intervals with exponent $\gamma - \varepsilon$.*

To finish the proof of the homogeneous case, we have now to study $\Lambda(d\varepsilon) = \nu(g \in d\varepsilon)$. In the following we denote by k an integer such that $\nu(s_k > 0) = 0$.

We remark that $\{g \geq \varepsilon\} \subset \{Card\{i, s_i > \varepsilon/k\} \geq 2\}$, so $h(\varepsilon/k) \geq \nu(g \geq \varepsilon)$. We notice also that $h(\varepsilon) \leq k\nu(g \geq \varepsilon \text{ or } d \leq 1 - \varepsilon)$. As $\nu(d \leq 1 - \varepsilon) \leq C\nu(g \geq \varepsilon)$ we get

$$\frac{h(\varepsilon)}{(C+1)k} \leq \nu(g \geq \varepsilon) \leq h(\varepsilon/k).$$

Using that h is regularly varying as $\varepsilon \to 0+$ with index $-\beta$, an easy calculus proves that $\sup\{\alpha > 0, \lim_{\lambda \to \infty} \lambda^{-\alpha}\Phi(\lambda) = \infty\} = \beta$ and so σ^{-1} is Holder-continuous with exponent $\beta - \varepsilon$ for all $\varepsilon > 0$. Hence we get that for each $t > 0$, $\dim_{\mathcal{H}} K(t) = \beta$ a.s. As for $t < s$, $K(t) \subset K(s)$, $\dim_{\mathcal{H}} K(t)$ increases with t, and so we have also $\dim_{\mathcal{H}} K(t) = \beta$ for all $t > 0$ simultaneously a.s.

It remains now to prove the lower bound for an i-fragmentation with strictly positive index of self similarity . Let us use now Proposition 4.8 which changes the index of self-similarity of

a fragmentation. Let $(U^\alpha(t), t \geq 0)$ be a self similar fragmentation fulfilling (H). We write $U^\alpha(t) = U(T_t^\alpha)$ as in Proposition 4.8 where $(U(t), t \geq 0)$ is a homogeneous fragmentation. We denote by $(g_t, t \geq 0)$ (resp. $(g_t^\alpha, t \geq 0)$) the left bound of the largest interval component of $U(t)$ (resp. $U^\alpha(t)$). We know that for all $T > 0$, $\dim_{\mathcal{H}}\{g_t, 0 \leq t \leq T\} \geq \beta$. Or for t small enough, we have $g_t^\alpha = g_{f(t)}$ where f is a continuous increasing function, so for all $t > 0$, there exists $t' > 0$ such that

$$\dim_{\mathcal{H}}(K^\alpha(s), 0 \leq s \leq t) \geq \dim_{\mathcal{H}}(g_s^\alpha, 0 \leq s \leq t) \geq \dim_{\mathcal{H}}(g_s, 0 \leq s \leq t') \geq \beta. \; \square$$

Corollary 5.7 *Let ν be a dislocation measure fulfilling Hypothesis 5.2. Let $(U(t), t \geq 0)$ be a self-similar i-fragmentation with characteristics $(\nu, 0, 0, \alpha)$ with $\alpha > 0$. Let $K(t) = [0,1] \backslash U(t)$. Then the packing dimension of $K(t)$ is β for all $t > 0$ simultaneously, a.s.*

Proof. Let us first recall the definition of the packing dimension [19]. For a subset $E \subset \mathbb{R}$ and $\alpha > 0$, let us define

$$M_\alpha(E) = \lim_{\varepsilon \to 0+} \sup \left\{ \sum_{i=1}^{\infty} (2r_i)^\alpha, \; [x_i - r_i, x_i + r_i] \text{ disjoint}, x_i \in E, r_i < \varepsilon \right\},$$

and

$$\widehat{M}_\alpha(E) = \inf \left\{ \sum_{n=1}^{\infty} M_\alpha(E_n), \; E \subseteq \bigcup_{n=1}^{\infty} E_n \right\}.$$

The packing dimension of E is defined by

$$\dim_\wp(E) = \inf\{\alpha > 0, \; \widehat{M}_\alpha(E) = 0\} = \sup\{\alpha > 0, \; \widehat{M}_\alpha(E) = \infty\}.$$

For a subset $E \subset \mathbb{R}$ and $\varepsilon > 0$, let $Z(E, \varepsilon)$ be the smallest number of interval of lengths 2ε needed to cover E. We define

$$\Delta(E) = \limsup_{\varepsilon \to 0} \frac{\log Z(E, \varepsilon)}{-\log \varepsilon}.$$

Tricot [19] proved that we have :

$$dim_\wp(E) = \inf \left\{ \sup_n \Delta(E_n), \; E \subset \cup_n E_n \right\}.$$

It is then easy to see that for all $E \subset \mathbb{R}$, we have $\dim_{\mathcal{H}} E \leq \dim_\wp E$. Hence, to prove Corollary 5.7, we just have to get an upper bound of the packing dimension of $K(t)$. We use the same idea as for the Hausdorff dimension. Let $I_\varepsilon =]0,1[\backslash \{$ interval components of $U(t)$ which size is larger than $\varepsilon\}$. So we have $K(t) \subset I_\varepsilon$ and $|I_\varepsilon| = M(\varepsilon, t)$ since ν is conservative. Furthermore, I_ε has at most $N(\varepsilon, t) + 1$ interval components. We deduce that

$$Z(K(t), \varepsilon) \leq Z(I_\varepsilon, \varepsilon) \leq h(\varepsilon) \left(\frac{M(\varepsilon, t)}{2\varepsilon h(\varepsilon)} + \frac{N(\varepsilon, t) + 1}{h(\varepsilon)} \right).$$

We get

$$\dim_\wp(K(t)) \leq \Delta(K(t)) \leq \limsup_{\varepsilon \to 0} \frac{\log h(\varepsilon)}{-\log \varepsilon} = \beta.$$

Hence, the packing dimension of the subset $K(t)$ coincides almost surely with its Hausdorff dimension (such subset is called "regular subset"). $\square$

To conclude this section, let us discuss an example. We consider the m-fragmentation introduced by Aldous and Pitman [1] to study the standard additive coalescent. Bertoin [6] gave a construction of an i-fragmentation $(U(t), t \geq 0)$ whose projection on $\mathcal{S}^{\downarrow}$ is this fragmentation. More precisely, let $\varepsilon = (\varepsilon_s, s \in [0,1])$ be a standard positive Brownian excursion. For every $t \geq 0$, we consider

$$\varepsilon_s^{(t)} = ts - \varepsilon_s, \qquad S_s^{(t)} = \sup_{0 \leq u \leq s} \varepsilon_u^{(t)}.$$

We define $U(t)$ as the constancy intervals of $(S_s^{(t)}, 0 \leq s \leq 1)$. Bertoin [7] proved also that $(|U(t)|^{\downarrow}, t \geq 0)$ is an m-fragmentation with index of self similarity $1/2$ and its dislocation measure is carried by the subset of sequences

$$\{s = (s_1, s_2, \dots) \in \mathcal{S}^{\downarrow}, s_1 = 1 - s_2 \text{ and } s_i = 0 \text{ for } i \geq 3\}$$

and is given by

$$\tilde{\nu}_{AP}(s_1 \in dx) = \left(2\pi x^3 (1-x)^3\right)^{-1/2} dx.$$

This proves that (H1), (H2) and (H3) hold with $\beta = 1/2$. Besides, as

$$\lim_{s \to 0^+} \frac{\varepsilon_s}{s} = \infty \ a.s.,$$

0 is almost surely an isolated point of $[0,1] \backslash U(t)$ and this implies that $\nu_{AP}(g > 0)$ is finite. Hence we have $\limsup \frac{\nu(g \geq \varepsilon)}{\nu(d \leq 1-\varepsilon)} < \infty$ and Hypothesis (H3) holds. By Theorem 5.3, we deduce that the Hausdorff dimension of $[0,1] \backslash U(t)$ is $\frac{1}{2}$ a.s., a fact that can be checked directly using properties of Brownian motion.

6 Interval components in uniform random order

Definition 6.1 *Let $\tilde{\nu}$ be a measure on $\mathcal{S}^{\downarrow}$ such that $\tilde{\nu}(\sum_i s_i < 1) = 0$. We define $\widehat{\nu}$ as the measure on $\mathcal{U}$ which projection on $\mathcal{S}^{\downarrow}$ is $\tilde{\nu}$ and which interval components are in uniform random order. More precisely, set $s = (s_i)_{i \in \mathbb{N}} \in \mathcal{S}^{\downarrow}$ with law $\tilde{\nu}$. Let $(V_i)_{i \in \mathbb{N}}$ be iid random variables uniform on $[0,1]$. We denote then U the random open subset of $]0,1[$ such that, if the decomposition of U in disjoint open intervals ranked by their length is $\coprod_{i=1}^{\infty} U_i$, we have*

- *For all $i \in \mathbb{N}$, $|U_i| = s_i$.*
- *For all $i \neq j$, $U_i \prec U_j \Leftrightarrow V_i \leq V_j$.*

Since we have $\sum_i s_i = 1$ a.s., there exists almost surely a unique open subset of $]0,1[$ fulfilling this two points. We denote by $\widehat{\nu}$ the distribution of U.

Proposition 6.2 *Let $(U(t), t \geq 0)$ is an i-fragmentation with measure $(\nu, 0, 0)$ and such that for all $t \geq 0$, $U(t)$ has interval components in uniform random order. Then ν has also interval components in uniform random order.*

Proof. Let $(F(t), t \geq 0)$ be the projection of $(U(t), t \geq 0)$ on $\mathcal{S}^{\downarrow}$. We know that F is then a m-fragmentation with measure $(\tilde{\nu}, 0)$ where $\tilde{\nu}$ is the image of ν by the canonical projection $\mathcal{U} \to \mathcal{S}^{\downarrow}$. Let $\gamma \in \mathcal{C}_n$. Let $\pi \in \mathcal{P}_n$ be the image of γ by the canonical projection $\wp_1$ between $\mathcal{C}$ and $\mathcal{P}_{\infty}$. Let now remark that we have

$$q_{\gamma} = \frac{1}{s} \lim_{s \to 0} \mathbb{P}(\Gamma_{[n]}(s) = \gamma) = \frac{1}{k!} q_{\pi},$$

where k is the number of blocks of γ and q_π the jump rate of the p-fragmentation. Let $\widehat{\nu}$ be the measure on $\mathcal{U}$ obtained in Definition 6.1 from ν. Let us recall that $\mathcal{Q}_{\infty,\gamma} = \{\gamma' \in \mathcal{C}, \gamma'_{[n]} = \gamma\}$ and define also $\mathcal{P}_{\infty,\pi} = \{\pi' \in \mathcal{P}_\infty, \pi'_{[n]} = \pi\}$. We have then

$$P^{\widehat{\nu}}(\mathcal{Q}_{\infty,\gamma}) = \frac{1}{k!} P^{\tilde{\nu}}(\mathcal{P}_{\infty,\pi}) = \frac{1}{k!} q_\pi = q_\gamma = P^{\nu}(\mathcal{Q}_{\infty,\gamma}).$$

So we get that $\nu = \widehat{\nu}$ and hence ν has interval components in uniform random order. $\square$

Let us notice that the proof uses $q_\gamma = \frac{1}{k!} q_\pi$, so if we want to extend this proposition to the time-inhomogeneous case, we must not only suppose that $U(t)$ has interval components in uniform random order, but more generally that the semi-group of $U(t)$, $q_{t,s}(]0,1[)$ has interval components in uniform random order for all $t \geq 0$ and for all $s > t$.

Conversely, we can ask if $(U(t), t \geq 0)$ is an i-fragmentation with measure $(\nu, 0, 0)$ and ν has interval components in uniform random order, does this implies that $U(t)$ has interval components in uniform random order ? The answer is clearly negative. Indeed, let ν be the following measure :

$$\nu = \delta_{U_1} + \delta_{U_2} \text{ with } U_1 = \left]0, \frac{1}{3}\right[\cup \left]\frac{1}{3}, \frac{2}{3}\right[\cup \left]\frac{2}{3}, 1\right[\text{ and } U_2 = \left]0, \frac{1}{2}\right[\cup \left]\frac{1}{2}, 1\right[.$$

Then ν has interval components in uniform random order, but $U(t)$ has not this property since we have

$$\mathbb{P}\left(U(t) = \left]0, \frac{1}{3}\right[\cup \left]\frac{1}{3}, \frac{1}{2}\right[\cup \left]\frac{1}{2}, \frac{2}{3}\right[\cup \left]\frac{2}{3}, 1\right[\right) > 0$$

and

$$\mathbb{P}\left(U(t) = \left]0, \frac{1}{6}\right[\cup \left]\frac{1}{6}, \frac{1}{2}\right[\cup \left]\frac{1}{2}, \frac{5}{6}\right[\cup \left]\frac{5}{6}, 1\right[\right) = 0.$$

6.1 Ruelle's fragmentation

In this section, we specify the semi-group of Ruelle's fragmentation seen as an interval fragmentation. Let us recall the construction of this interval fragmentation [10].

Let $(\sigma^*_t, 0 < t < 1)$ be a family of stable subordinators such for every $0 < t_n < \ldots < t_1 < 1$, $(\sigma^*_{t_1}, \ldots, \sigma^*_{t_n}) \overset{law}{=} (\sigma_{t_1}, \ldots, \sigma_{t_n})$ where $\sigma_{t_i} = \tau_{\alpha_1} \circ \ldots \circ \tau_{\alpha_i}$ and $(\tau_{\alpha_i}, 1 \leq i \leq n)$ are n independent stable subordinators with indices $\alpha_1, \ldots, \alpha_n$ such that $t_i = \alpha_1 \ldots \alpha_i$. Fix $t_0 \in]0,1[$ and for $t \in]t_0, 1[$ define T_t by :

$$\sigma^*(T_t) = \sigma^*_{t_0}(1).$$

Then consider the open subset :

$$U(t) = \left]0, 1\right[\setminus \left\{ \frac{\sigma^*_t(u)}{\sigma^*_{t_0}(1)}, 0 \leq u \leq T_t \right\}^{cl}.$$

Bertoin and Pitman proved that $(U(t), t \in [t_0, 1[)$ is an i-fragmentation (with initial state $U(t_0) \neq \mathbf{1}$ a.s.) and the semi-group of transition at time t to time s of the m-fragmentation $(|U(t)|^{\downarrow}, t \in [t_0, 1[)$ is $PD(s, -t)$-FRAG where $PD(s, -t)$ denotes the Poisson-Dirichlet law with parameter $(s, -t)$ (see [17] for more details about the Poisson-Dirichlet laws). Furthermore, the instantaneous dislocation measure of this m-fragmentation at time t is $\frac{1}{t} PD(t, -t)$ (cf. [2]). We would like now to calculate the dislocation measure of the i-fragmentation $(U(t), t \in [t_0, 1[)$.

Lemma 6.3 *Let us define $\widehat{PD}(t,0)$ as the measure on $\mathcal{U}$ obtained from $PD(t,0)$ by Definition 6.1. The distribution at time t of $U(t)$ is $\widehat{PD}(t,0)$.*

Proof. For $t \in]t_0, 1[$, we have $\sigma^*_{t_0} = \sigma^*_t \circ \tau_\alpha$ where $\alpha t = t_0$ and τ_α is a stable subordinator with index α and independent of σ^*_t. Hence we get

$$U(t) = \Big]0,1\Big[\Big\backslash \left\{ \frac{\sigma^*_t(u)}{\sigma^*_t(\tau_{\alpha(1)})}, 0 \leq u \leq \tau_\alpha(1) \right\}^{cl}.$$

We can thus write

$$U(t) = \Big]0,1\Big[\Big\backslash \left\{ \frac{\sigma_t(x)}{\sigma_t(a)}, x \in [0,a[\right\}^{cl},$$

where σ_t is a stable subordinator with index t and a is a random variable independent of σ_t. If we denote by $(t_i, s_i)_{i\geq 1}$ the time and size of the jump of σ_t in the interval $[0,a[$ ranked by decreasing order of the size of the jumps, this family has the same law of $(t_{\tau(i)}, s_i)_{i\geq 1}$ for any τ permutation of $\mathbb{N}$. □

Proposition 6.4 *The semi-group of transition of the Ruelle's interval fragmentation from time t to time s is $\widehat{PD}(s,-t)$-FRAG and the instantaneous dislocation measure at time t is $\frac{1}{t}\widehat{PD}(t,-t)$.*

We would like now to apply Proposition 6.2 to determine the instantaneous measure of dislocation of Ruelle's fragmentation, but this proposition holds only for time-homogeneous fragmentation. If the fragmentation is inhomogeneous in time, we must first prove that the semi-group of $U(t)$ has interval component in uniform order. Fix $t \geq 0$ and $s > t$. Fix $y \in]0,1[$ and denote by $I(t)$ the interval component of $U(t)$ containing y. We shall prove that $U(s) \cap I(t)$ has its interval component in uniform random order. By the construction of $U(t)$, there exists $x \in]0, T_t[$ such that

$$I(t) = \Big] \frac{\sigma^*_t(x^-)}{\sigma^*_{t_0}(1)}, \frac{\sigma^*_t(x)}{\sigma^*_{t_0}(1)} \Big[.$$

We have $\sigma^*_t = \sigma^*_s \circ \tau_{t/s}$ where $\tau_{t/s}$ is a stable subordinator with index t/s and is independent of σ^*_{t+s}. Hence, we get :

$$U(s) \cap I(t) = I(t) \Big\backslash \left\{ \frac{\sigma^*_s(y)}{\sigma^*_{t_0}(1)},\ \tau_{t/s}(x^-) \leq y \leq \tau_{t/s}(x) \right\}^{cl}.$$

Since $\tau_{t/s}$ is independent of σ^*_s, the jump of σ^*_s on the interval $]\tau_{t/s}(x^-), \tau_{t/s}(x)[$ are in uniform random order. Since as m-fragmentation the semi-group of transition is $PD(s,-t)$-FRAG, we deduce that, as i-fragmentation, the semi-group is $\widehat{PD}(s,-t)$-FRAG. To prove that the dislocation measure at time t is $\frac{1}{t}\widehat{PD}(t,-t)$, we just have to apply the Proposition 6.2. □

6.2 Dislocation measure of the fragmentation derived from the additive coalescent

Recall the construction of an i-fragmentation $(U(t), t \geq 0)$ from a Brownian motion exposed in Section 5. We already know its characteristics as a m-fragmentation : the erosion rate is null, the index of self similarity is equal to 1/2 and the dislocation measure $\tilde{\nu}_{AP}$ is given by :

$$\tilde{\nu}_{AP}(s_1 \in dx) = (2\pi x^3(1-x^3))^{-1/2} dx \quad \text{for } x \geq 1/2, \qquad \tilde{\nu}_{AP}(s_1 = 1 - s_2) = 1.$$

Proposition 6.5 *The i-fragmentation derived from a Brownian motion [6] has dislocation measure ν_{AP} such that :*

- *ν_{AP} is carried by the subset of]0,1[shaped as $]0,1[\backslash x$. So we will write $\nu_{AP}(x)$ instead of $\nu_{AP}(]0,1[\backslash x)$.*
- *For all $x \in]0,1[, \quad \nu_{AP}(dx) = (2\pi x(1-x^3))^{-1/2}dx$.*

Notice that we have $\nu_{AP}(dx) = x\tilde{\nu}_{AP}(s_1 \in dx \text{ or } s_2 \in dx)$ for all $x \in]0,1[$. Hence, given that the m-fragmentation splits in two block of size x and $1-x$, the left block of the i-fragmentation will be a size biased pick of x and $1-x$.

Proof. The first part of the proposition is immediate since we have $\tilde{\nu}_{AP}(s_1 = 1 - s_2) = 1$. For the second part, let us use Theorem 9 in [6] which gives the distribution ρ_t of the most left fragment of $U(t)$:

$$\rho_t(dx) = t\frac{1}{\sqrt{2\pi x(1-x)^3}}\exp\left(-\frac{xt^2}{2(1-x)}\right)dx \quad \text{for all } x \in]0,1[.$$

We get

$$\nu_{AP}(dx) = \lim_{t\to 0}\frac{1}{t}\rho_t(dx) = \frac{1}{\sqrt{2\pi x(1-x)^3}}. \square$$

We can also give a description of the distribution at time $t > 0$ of $U(t)$. Recall the result obtained by Chassaing and Janson [12]. For a random process X on $\mathbb{R}$ and $t \geq 0$, we define $\ell_t(X)$ as the local time of X at level 0 on the interval $[0,t]$, i.e.

$$\ell_t(X) = \lim_{\varepsilon\to 0+}\frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{\{|X_s|<\varepsilon\}}ds,$$

whenever the limit makes sense.

Let X^t be a reflected Brownian bridge conditioned on $\ell_1(X^t) = t$. We define $\mu \in]0,1[$ such that

$$\ell_\mu(X^t) - t\mu = \max_{0\leq u\leq 1}\ell_u(X^t) - tu.$$

It is well known that this equation has almost surely a unique solution. Let us define the process $(Z^t(s), 0 \leq s \leq 1)$ by

$$Z^t(s) = X^t(s + \mu\,[\text{mod } 1]).$$

Chassaing and Janson [12] have proved that for each $t \geq 0$

$$U(t) \overset{law}{=}]0,1[\backslash\{x \in [0,1], Z^t(x) = 0\}.$$

Besides, as the inverse of the local time of X^t defined by

$$T_x = \inf\{u \geq 0,\ \ell_u(X^t) > x\}$$

is a stable subordinator with Lévy measure $(2\pi x^3)^{-1/2}dx$ conditioned to $T_t = 1$, we deduce the following description of the distribution of $U(t)$:

Corollary 6.6 *Let $t > 0$. Let T be a stable subordinator with Lévy measure $(2\pi x^3)^{-1/2}dx$ conditioned to $T_t = 1$. Let us define m as the unique number on $[0,t]$ such that*

$$tT_{m^-} - m \leq tT_u - u \quad \textit{for all } u \in [0,t],$$

where $T_{m^-} = \lim_{x\to m^-} T_x$. We set :

$$\begin{aligned} \tilde{T}_x &= T_{m+x} - T_{m^-} & \textit{for } 0 < x < t-m, \\ & T_{m+x-t} - T_{m^-} + 1 & \textit{for } t-m \leq x \leq t. \end{aligned}$$

Then

$$U(t) \overset{law}{=}]0,1[\setminus \{\tilde{T}_x, x \in [0,t]\}^{cl}.$$

Proof. It is clear that $\{u, X^t(u) = 0\}$ coincides with $\{T_x, x \in [0,t]\}^{cl}$ when T is the inverse of the local time of X^t. Hence, we just have to check that if we set $m = \ell_\mu(X^t)$, then m verifies the equation $tT_{m^-} - m \leq tT_u - u$ for all $u \in [0,t]$. Since $X^t(\mu) = 0$, we have $T_{m^-} = \mu$, thus we get :

$$tT_{m^-} - m = t\mu - \ell_\mu(X^t) \leq tv - \ell_v(X^t) \text{ for all } v \in [0,1].$$

Let us fix $u \in [0,t]$. Since $\ell_v(X^t)$ is a continuous function, there exists $v \in [0,1]$ such that $\ell_v(X^t) = u$. Besides we have $T_u^- \leq v \leq T_u$, so we get

$$tT_{m^-} - m \leq tT_u - u. \square$$

Hence, the distribution of $[0,1] \setminus U(t)$ can be obtained as the closure of the range of a stable subordinator $(T_s, 0 \leq s \leq t)$ with index $1/2$ and conditioned on $T_t = 1$ randomly shifted (recall also that Chassaing and Jason [12] have proved that the left most fragment of $U(t)$ is size-biased picked).

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